Int. J. Solids Structures. 1975, Vol. 11, pp. 1079–1096. Pergamon Press. Printed in Gt. Britain.

STRESS ANALYSIS FOR A LAYER OF FINITE LENGTH BONDED TO A HALF-SPACE OF IDENTICAL MATERIAL

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(Received 29 April 1974; revised 10 February 1975)

Abstract—The problem in the plane theory of elasticity of an elastic layer bonded to an elastic half-space of the same material is considered. The formulation is achieved by means of integral transforms and the problem is reduced to the solution of a system of singular integral equations. A numerical solution is accomplished for the half-space and layer by use of the collocation scheme developed by Erdogan and Gupta.

INTRODUCTION

The problems that will be considered in this paper are concerned with the bonding of two elastic bodies to one another. For the considered problem the bond is of such a nature that the thickness of the adhesive can be considered negligible; therefore continuity of displacements between the bodies will be assumed. Of interest are the magnitudes and distributions of the normal and shear stresses in the bond since they will determine the strength of the bond, and if it fails, the manner of failure. The specific geometry to be considered is that for a layer of finite length bonded to a half-space (Fig. 1a). The cases studied are within the context of the linear theory of elasticity where dynamic and temperature effects are neglected, and the two bodies are of identical materials.

Considerable research into the nature of adhesive bonding has been done experimentally and by means of engineering theories, e.g. beam or shell theories. The use of the engineering theories allows problems of fairly high complexity to be studied, but the complete determination of bond stress would be only approximately given. On the other hand an elasticity theory can determine the bond stress with greater accuracy, but the geometric complexity of the problem must be diminished. A general survey of the field is provided in the book edited by Eley[1] which reviews both the analytical and experimental theories. In particular, an article by Sneddon discusses some of the approximate theories. The strength of lap joints has also been discussed by Greenwood[2], who drew some conclusions as to the influence of the joint geometry upon its strength. Some interesting solutions to particular problems using the engineering theory are given by Goland and Reissner[3] and Lubkin and Reissner[4].

The solutions which use the engineering theory do not have to apply special consideration to the end of the layer. On the other hand, solution to the problem of bonded bodies, a lap joint for example, by means of the theory of elasticity becomes very complicated; therefore, theory of elasticity solutions generally ignore the ends of the strip by considering the problem of infinite strips bonded together. Chang and Muki[5] deal with stress distributions in lap joints under cleavage and tension-shear loading within the context of theory of elasticity. Infinite layers are considered and the problem is reduced to Fredholm integral equations which were solved numerically. A related problem for two bonded layers was solved by Keer[6], and for a double lap joint by Keer and Chantaramungkorn[7], who reduced the governing equations to a system of

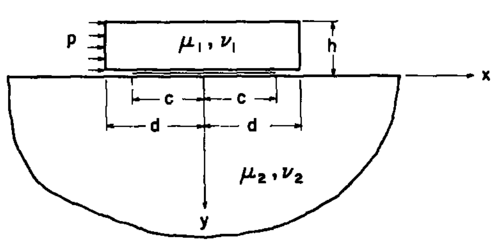

Fig. 1a. Geometry and coordinate system of a layer of finite thickness and length perfectly bonded to a half-space.

singular equations which were solved using the numerical scheme of Erdogan and Gupta[8]. Again it was assumed that the solution to the problem of infinite layers was a realistic approximation to that of layers with ends. It was assumed that the general behavior of the bond stresses and their effect could be adequately explained by ignoring the effects of the end of the layer. One goal of the present investigation is to compare the stress distribution of an infinite layer with that of a finite layer bonded to a half-space of the same material.

The problem to be studied is somewhat analogous to that of an elastic stiffener bonded to a half-space which was solved by Erdogan and Gupta[9]. The thickness of the stiffener was assumed to be sufficiently small as to allow it to be approximated by a beam in extension (generalized plane stress assumption) and only the bond shear stress was considered. However, in this case there will be no restrictive assumption on the covering layer. The numerical analysis is applicable only for the case where the cover layer is of the same material as the half-space.

GENERAL APPROACH

The present analysis considers the plane problem in the theory of elasticity for a layer of finite thickness, h, and length, $2d$, that is perfectly bonded to an elastic half-space. The length of the bond is $2c$, and its thickness is assumed to be zero in the analysis. The geometry, coordinate system, and loading configuration of the problem are given in Fig. 1a. The rectangular coordinate system, x and y, is oriented such that the plane $x = 0$ gives geometrical symmetry to the problem and the plane $y = 0$ is on the line of the bond. The elastic constants of the layer and half-space, shear modulus and Poisson's ratio, are denoted by μ_1, ν_1 and μ_2, ν_2, respectively. In the subsequent sections the superscript or subscript, (1) and (2), will refer to the layer and half-space, respectively. The layer is subjected to a uniform compressive load, p, on the surface $x = -d$, and is stress free on $y = -h$ and $x = d$.

The boundary and continuity conditions of the problem are

$$\tau_{yy}^{(1)} = \tau_{xy}^{(1)} = 0, \quad y = -h, \quad 0 \le |x| \le d \tag{1}$$

$$\left.\begin{aligned} \tau_{xx}^{(1)} &= -p \\ \tau_{xy}^{(1)} &= 0 \end{aligned}\right\}, \quad x = -d, \quad -h < y < 0 \tag{2}$$

$$\tau_{xx}^{(1)} = \tau_{xy}^{(1)} = 0, \quad x = d, \quad -h < y < 0 \tag{3}$$

$$\left.\begin{aligned} \tau_{xy}^{(1)} &= \tau_{xy}^{(2)} \\ \tau_{yy}^{(1)} &= \tau_{yy}^{(2)} \end{aligned}\right\}, \quad y = 0, \quad 0 \le |x| < d \tag{4}$$

$$\tau_{xy}^{(2)} = \tau_{yy}^{(2)} = 0, \quad y = 0, \quad c < |x| < \infty \tag{5}$$

$$\left.\begin{aligned} u_x^{(1)} - u_x^{(2)} &= 0 \\ u_y^{(1)} - u_y^{(2)} &= 0 \end{aligned}\right\}, \quad y = 0, \quad 0 \le |x| \le c. \tag{6}$$

Here, eqns (1)–(3) describe the loading configuration on the surfaces of the layer. The continuity of the shear and normal stresses on the plane $y = 0$ is given in eqns (4). The mixed boundary conditions on the plane $y = 0$, eqns (5) and (6), express the fact that the displacements are continuous in the bond region, $-c \le x \le c$, and that the shear and normal stresses vanish exterior to the bond.

The general method for solving the given problem will be by means of a superposition technique where the solution is written as the sum of two solutions in the form:

$$U^{(j)} = \bar{u}^{(j)} + u^{(j)}, \quad j = 1, 2. \tag{7}$$

Here, $U^{(j)}$ is the required solution and $\bar{u}^{(j)}$, $u^{(j)}$ are solutions to two problems, the superposition of which will satisfy the boundary and continuity conditions of the desired problem.

The two superposition solutions required are given by Figs. 1b and 1c and are the problems of an infinite layer bonded to an elastic half-space (Fig. 1b) and of a layer having cracks perpendicular to its surfaces (Fig. 1c). The superposition solutions will be referred to as I and II, respectively, referring to Figs. 1b and 1c.

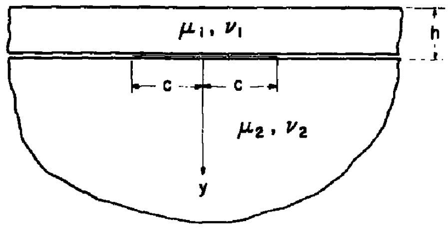

Fig. 1b. Geometry and coordinate system of an infinite layer bonded to a half-space.

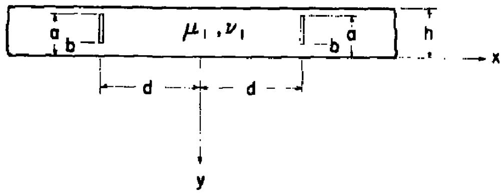

Fig. 1c. Geometry and coordinate system of a layer having cracks perpendicular to its surfaces.

The boundary and continuity conditions for problem I are the following:

$$\tau_{xy}^{(1)} = \tau_{yy}^{(1)} = 0, \quad y = -h, \quad 0 \le |x| < \infty \tag{8}$$

$$\left.\begin{aligned} \tau_{xy}^{(1)} &= \tau_{xy}^{(2)} \\ \tau_{yy}^{(1)} &= \tau_{yy}^{(2)} \end{aligned}\right\}, \quad y = 0, \quad 0 \le |x| < \infty \tag{9}$$

$$\tau_{xy}^{(2)} = \tau_{yy}^{(2)} = 0, \quad y = 0, \quad c < |x| < \infty \tag{10}$$

$$\left.\begin{aligned} u_x^{(1)} - u_x^{(2)} &= \frac{\kappa_1 + 1}{4\mu_1} f_1(x) \\ u_y^{(1)} - u_y^{(2)} &= \frac{\kappa_1 + 1}{4\mu_1} f_2(x) \end{aligned}\right\}, \quad y = 0, \quad 0 \le |x| \le c \tag{11}^{\dagger}$$

where $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for the plane stress and where the right-hand sides of eqns (11) are displacement mismatches in the bond region on the plane $y = 0$.

The problem I is seen to have the same boundary and continuity conditions as the desired problem except for the condition that the stresses on $|x| = d$ of the layer and the displacement continuity in the bond region are not satisfied. Thus, the necessity for developing a solution for problem II is seen. The relevant physical quantities of problem I that are important for the superposition technique are the normal and shear stresses calculated on the surfaces $|x| = d$ of the layer. They are

$$\tau_{xx}^{(1)}(-d, y) = \sigma_1(y) \tag{12}$$

$$\tau_{xy}^{(1)}(-d, y) = \tau_1(y) \tag{13}$$

$$\tau_{xx}^{(1)}(d, y) = \sigma_2(y) \tag{14}$$

$$\tau_{xy}^{(1)}(d, y) = \tau_2(y) \tag{15}$$

where $\sigma_1, \tau_1, \sigma_2$ and τ_2 are functions of y used to describe these stresses.

The problem II is that of a layer having two identical vertical cracks at a distance d from the center of the layer. Each crack is of length $a - b$ with the crack tips at distances a and b from the lower surface of the layer. The normal and shear stresses will be prescribed on the surfaces of each crack. Problem II has the following boundary conditions:

$$\tau_{xy}^{(1)} = \tau_{yy}^{(1)} = 0, \quad y = -h, \quad 0 \le |x| < \infty \tag{16}$$

$$\tau_{xy}^{(1)} = \tau_{yy}^{(1)} = 0, \quad y = 0, \quad 0 \le |x| < \infty \tag{17}$$

$^{\dagger}f_1(x)$ and $f_2(x)$ arise from possible superposition of elementary solutions[6, 7].

$$\left.\begin{aligned}\tau_{xx}^{(1)} &= p_1(y)\\ \tau_{xy}^{(1)} &= q_1(y)\end{aligned}\right\}, \quad x = -d, \quad -a < y < -b \tag{18}$$

$$\left.\begin{aligned}\tau_{xx}^{(1)} &= p_2(y)\\ \tau_{xy}^{(1)} &= q_2(y)\end{aligned}\right\}, x = d, \quad -a < y < -b. \tag{19}$$

Here, p_1, q_1, p_2 and q_2 are normal and shear stresses prescribed on the surfaces of the cracks. The important physical quantities to be computed are the horizontal and vertical displacements on the plane $y = 0$, given as

$$\left.\begin{aligned}u_x^{(1)} &= \frac{\kappa_1 + 1}{4\mu_1} g_1(x)\\ u_y^{(1)} &= \frac{\kappa_1 + 1}{4\mu_1} g_2(x)\end{aligned}\right\}, \quad y = 0, \quad 0 \le |x| < \infty \tag{20}$$

Here, g_1 and g_2 are functions of x used to describe these displacements.

It is now clear that the superposition of solutions I and II will give the desired solutions provided that the following conditions are satisfied:

$$\left.\begin{aligned}\sigma_1(y) + p_1(y) &= -p\\ \tau_1(y) + q_1(y) &= 0\end{aligned}\right\}, \quad x = -d, -a < y < -b \tag{21}$$

$$\left.\begin{aligned}\sigma_2(y) + p_2(y) &= 0\\ \tau_2(y) + q_2(y) &= 0\end{aligned}\right\}, x = d, \quad -a < y < -b \tag{22}$$

$$\left.\begin{aligned}f_1(x) + g_1(x) &= 0\\ f_2(x) + g_2(x) &= 0\end{aligned}\right\}, y = 0, \quad 0 \le |x| \le c. \tag{23}$$

Here, eqns (21) and (22) satisfy the boundary conditions, eqns (2) and (3), on the surfaces $|x| = d$ of the layer under the condition that the limiting value of b approaches zero and that of a approaches h. Equations (23) satisfy the mixed boundary conditions, eqns (6), for displacements.

Solution to Problem I

Solutions satisfying the displacement field equations appropriate for a layer and half-space can be written as exponential transforms (see, e.g. Keer and Chantaramungkorn [7]).

In view of the continuity conditions, eqns (9), and the boundary conditions of the top surface of the layer, eqns (8), the integral transforms can be written in terms of the shear and normal bond stresses $\tau(x)$ and $\sigma(x)$ and when put into the differentiated form of the displacement boundary conditions, eqns (11), lead immediately to the following pair of coupled singular integral equations (see, e.g. eqns (17)–(21) in [7] with $\beta_2 \to 0$):

$$\frac{1}{\pi}\int_{-c}^{c} \frac{\tau(s)\,\mathrm{d}s}{s - x} + \alpha_1\sigma(x) + \int_{-c}^{c} [\tau(s)K_1(s, x) + \sigma(s)K_2(s, x)]\,\mathrm{d}s = \frac{(1 + \alpha_2)}{2}\frac{\mathrm{d}f_1}{\mathrm{d}x}, \quad 0 \le |x| \le c \tag{24a}$$

$$\frac{1}{\pi}\int_{-c}^{c} \frac{\sigma(s)\,\mathrm{d}s}{s - x} - \alpha_1\tau(x) - \int_{-c}^{c} [\tau(s)K_2(s, x) - \sigma(s)K_3(s, x)]\,\mathrm{d}s = \frac{(1 + \alpha_2)}{2}\frac{\mathrm{d}f_2}{\mathrm{d}x}, \quad 0 \le |x| \le c \tag{24b}$$

where

$$K_1(s, x) = \frac{(1 + \alpha_2)}{2\pi}\int_0^\infty \frac{1}{\Delta}(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta + \beta^2 - \beta)\sin\xi(s - x)\,\mathrm{d}\xi \tag{25a}$$

$$K_2(s, x) = \frac{(1 + \alpha_2)}{2\pi}\int_0^\infty \frac{1}{\Delta}\beta^2\cos\xi(s - x)\,\mathrm{d}\xi \tag{25b}$$

$$K_3(s, x) = \frac{(1 + \alpha_2)}{2\pi}\int_0^\infty \frac{1}{\Delta}(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta + \beta^2 + \beta)\sin\xi(s - x)\,\mathrm{d}\xi \tag{25c}$$

$$\Delta = \mathrm{sh}^2\beta - \beta^2 \tag{26}$$

and

$$\beta = |\xi|h, \tag{27a}$$

$$\alpha_1 = \frac{\mu_2(\kappa_1 - 1) - \mu_1(\kappa_2 - 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)} \tag{28a}$$

$$\alpha_2 = \frac{\mu_2(\kappa_1 + 1) - \mu_1(\kappa_2 + 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)}. \tag{28b}$$

Constants α_1, α_2 were introduced by Dundurs[10]. In addition, the singular integral equations require the following subsidiary conditions:

$$\int_{-c}^{c} \tau(x)\,\mathrm{d}x = -ph \tag{29a}$$

$$\int_{-c}^{c} \sigma(x)\,\mathrm{d}x = 0 \tag{29b}$$

which are simply the conditions of static equilibrium in the bond region. The normal and shear stresses at any point on the layer are easily found to be

$$\tau_{xx}(x, y) = \frac{1}{\pi}\int_{-c}^{c} [\tau(s)M_1(s, x, y) + \sigma(s)M_2(s, x, y)]\,\mathrm{d}s \tag{30}$$

$$\tau_{xy}(x, y) = \frac{1}{\pi}\int_{-c}^{c} [\tau(s)M_3(s, x, y) + \sigma(s)M_4(s, x, y)]\,\mathrm{d}s \tag{31}$$

where $\zeta = \xi y$ and

$$M_1 = \frac{2(s-x)^3}{[y^2 + (s-x)^2]^2} + \int_0^\infty \frac{1}{\Delta}[(2\,\mathrm{ch}\,\zeta + \zeta\,\mathrm{sh}\,\zeta)(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta + \beta^2 - \beta) + (\mathrm{sh}\,\zeta + \zeta\,\mathrm{ch}\,\zeta)\beta^2]\sin\xi(s-x)\mathrm{d}\xi \tag{32a}$$

$$M_2 = -\frac{2y(s-x)^2}{[y^2 + (s-x)^2]^2} + \int_0^\infty \frac{1}{\Delta}[(2\,\mathrm{ch}\,\zeta + \zeta\,\mathrm{sh}\,\zeta)\beta^2 + (\mathrm{sh}\,\zeta + \zeta\,\mathrm{ch}\,\zeta)(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta + \beta^2 + \beta)]\cos\xi(s-x)\,\mathrm{d}\xi \tag{32b}$$

$$M_3 = -\frac{2y(s-x)^2}{[y^2 + (s-x)^2]^2} + \int_0^\infty \frac{1}{\Delta}[\zeta\,\mathrm{sh}\,\zeta\beta^2 + (\mathrm{sh}\,\zeta + \zeta\,\mathrm{ch}\,\zeta)(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta + \beta^2 - \beta)]\cos\xi(s-c)\,\mathrm{d}\xi \tag{32c}$$

$$M_4 = \frac{2y^2(s-x)}{[y^2 + (s-x)^2]^2} - \int_0^\infty \frac{1}{\Delta}[\zeta\,\mathrm{sh}\,\zeta(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta + \beta^2 + \beta) + (\mathrm{sh}\,\zeta + \zeta\,\mathrm{ch}\,\zeta)\beta^2]\sin\xi(s-x)\,\mathrm{d}\xi. \tag{32d}$$

The unknown stress functions σ_1, τ_1, σ_2 and τ_2 can be obtained by evaluating eqns (30) and (31) on $x = -d$ and d, respectively. For identical material $\alpha_1 = \alpha_2 = 0$.

Solution to Problem II

Problem II, which is shown in Fig. 1c, will be solved by a superposition technique. First, a solution to the problem of a crack in an infinitely extended body is given. In this problem, shown in Fig. 2, the vertical crack of length $a - b$ is located such that the crack is at a distance $-d$ from the y-axis and the tips of the crack are at distances $-a$ and $-b$ from the x-axis. Normal and shear stresses are prescribed on the faces of the crack.

If the jumps in displacement derivatives are given as

$$A(y) = \frac{\partial u_x^{(1)}}{\partial y} - \frac{\partial u_x^{(2)}}{\partial y}, \qquad x = -d \tag{33a}$$

$$B(y) = \frac{\partial u_y^{(1)}}{\partial v} - \frac{\partial u_y^{(2)}}{\partial y}, \qquad x = -d \tag{33b}$$

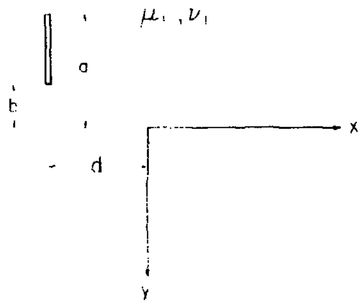

Fig. 2. Geometry and coordinate system of a crack in an infinitely extended body.

when $-a < y < -b$ and $A \equiv B \equiv 0$ otherwise, then it is easy to verify that general expressions for the stresses are

$$\tau_{xx} = \frac{2\mu}{\pi(1+\kappa)}\left\{-\int_a^b A(t)(t-y)[(t-y)^2+3(x+d)^2]\Delta_1\,dt + (x+d)\int_a^b B(t)[(t-y)^2-(x+d)^2]\Delta_1\,dt\right\} \quad (34a)$$

$$\tau_{xy} = \frac{2\mu}{\pi(1+\kappa)}\left\{(x+d)\int_a^b A(t)[(t-y)^2-(x+d)^2]\Delta_1\,dt - \int_a^b B(t)(t-y)[(t-y)^2-(x+d)^2]\Delta_1\,dt\right\} \quad (34b)$$

$$\tau_{yy} = \frac{2\mu}{\pi(1+\kappa)}\left\{\int_a^b A(t)(t-y)[(t-y)^2-(x+d)^2]\Delta_1\,dt - (x+d)\int_a^b B(t)[3(t-y)^2+(x+d)^2]\Delta_1\,dt\right\} \quad (34c)$$

where

$$\Delta_1 = \Delta_1(t,x,y) = [(t-y)^2+(x+d)^2]^{-2}. \quad (35)$$

In eqns (33) the superscript 1 refers to the region to the left of the crack where $x < -d$ and the superscript 2 refers to that to the right of the crack where $x > -d$. The expressions for stresses, eqns (34) can be applied for any value of x and y.

The significant physical quantities, $\partial u_x/\partial x$ and $\partial u_y/\partial x$ on $y=0$, can be written as

$$\frac{\partial u_x}{\partial x} = \frac{1}{2\pi(1+\kappa)}\left\{-\int_a^b A(t)t[(\kappa-1)t^2+(\kappa+3)(x+d)^2]\Delta_3\,dt + (x+d)\int_a^b B(t)[(5-\kappa)t^2-(\kappa-1)(x+d)^2]\Delta_3\,dt\right\} \quad (36a)$$

$$\frac{\partial u_y}{\partial x} = \frac{1}{2\pi(1+\kappa)}\left\{(x+d)\int_a^b A(t)[(\kappa+3)t^2+(\kappa-1)(x+d)^2]\Delta_3\,dt - \int_a^b B(t)t[(\kappa+3)t^2+(\kappa-1)(x+d)^2]\Delta_3\,dt\right\} \quad (36b)$$

where

$$\Delta_3 = \Delta_1(t,x,0). \quad (37)$$

The solutions for the problem where the crack is located at a distance d from the y-axis can be obtained by replacing d by $-d$, $A(t)$ and $B(t)$ by $C(t)$ and $D(t)$ and Δ_1 by Δ_2, respectively, where

$$C(y)=\frac{\partial u_x^{(1)}}{\partial y}-\frac{\partial u_x^{(2)}}{\partial y}, \qquad x=d \tag{38a}$$

$$D(y)=\frac{\partial u_y^{(1)}}{\partial y}-\frac{\partial u_y^{(2)}}{\partial y}, \qquad x=d \tag{38b}$$

$$\Delta_2\equiv\Delta_2(t,x,y)=[(t-y)^2+(x-d)^2]^{-2}. \tag{39}$$

In this case the superscript 1 refers to the region to the left of the crack where $x<d$ and the superscript 2 refers to that to the right of the crack where $x>d$.

The superposition of the solutions to these two problems provides shear and normal stresses on $y=-h$ and $y=0$. Next, a layer solution with prescribed stresses on the surfaces $y=-h$ and $y=0$ is developed to cancel those from the parallel crack solutions. Such a solution can also be taken from [7] by suitable reduction. The layer solution with cracks yields the following results which are appropriate to the present analysis:

$$\begin{aligned}\tau_{xx}(x,y)=\frac{4\mu}{\pi(1+\kappa)}\Big[\Big[\int_{-a}^{-b}A(t)\int_0^\infty N_1\cos\xi(x+d)\,\mathrm{d}\xi\,\mathrm{d}t\\+\int_{-a}^{-b}B(t)\int_0^\infty N_2\sin\xi(x+d)\,\mathrm{d}\xi\,\mathrm{d}t+\int_{-a}^{-b}C(t)\int_0^\infty N_1\cos\xi(x-d)\,\mathrm{d}\xi\,\mathrm{d}t\\+\int_{-a}^{-b}D(t)\int_0^\infty N_2\sin\xi(x-d)\,\mathrm{d}\xi\,\mathrm{d}t\Big]\end{aligned} \tag{40a}$$

$$\begin{aligned}\tau_{xy}(x,y)=\frac{4\mu}{\pi(1+\kappa)}\Big[\Big[\int_{-a}^{-b}A(t)\int_0^\infty N_3\sin\xi(x+d)\,\mathrm{d}\xi\,\mathrm{d}t\\+\int_{-a}^{-b}B(t)\int_0^\infty N_4\cos\xi(x+d)\,\mathrm{d}\xi\,\mathrm{d}t+\int_{-a}^{-b}C(t)\int_0^\infty N_3\sin\xi(x-d)\,\mathrm{d}\xi\,\mathrm{d}t\\+\int_{-a}^{-b}D(t)\int_0^\infty N_4\cos\xi(x-d)\,\mathrm{d}\xi\,\mathrm{d}t\Big]\end{aligned} \tag{40b}$$

where

$$\begin{aligned}\Delta N_1=(2\,\mathrm{ch}\,\zeta+\zeta\,\mathrm{sh}\,\zeta)[\eta\,\mathrm{sh}\,\beta\,\mathrm{ch}\,(\beta+\eta)-\beta\eta\,\mathrm{ch}\,\eta+\mathrm{sh}\,\eta(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta+\beta^2-\beta)]\\+(\mathrm{sh}\,\zeta+\zeta\,\mathrm{ch}\,\zeta)[\eta\,\mathrm{sh}\,\beta\,\mathrm{sh}\,(\beta+\eta)+\beta(\beta+\eta)\,\mathrm{sh}\,\eta]\\+\frac{1}{2}\Delta\,\mathrm{e}^{\eta}[(2+\zeta)\,\mathrm{e}^{\zeta}-\eta\,\mathrm{e}^{-\zeta}]\end{aligned} \tag{41a}$$

$$\begin{aligned}\Delta N_2=(2\,\mathrm{ch}\,\zeta+\zeta\,\mathrm{sh}\,\zeta)[\beta(\beta+\eta)\,\mathrm{sh}\,\eta-\eta\,\mathrm{sh}\,\beta\,\mathrm{sh}\,(\beta+\eta)]\\+(\mathrm{sh}\,\zeta+\zeta\,\mathrm{ch}\,\zeta)[\mathrm{sh}\,\eta(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta+\beta^2+\beta)-\beta\eta\,\mathrm{ch}\,\eta\\-\eta\,\mathrm{sh}\,\beta\,\mathrm{ch}\,(\beta+\eta))]+\frac{1}{2}\Delta\,\mathrm{e}^{\eta}[(1+\zeta)\,\mathrm{e}^{\zeta}+\eta\,\mathrm{e}^{-\zeta}]\end{aligned} \tag{41b}$$

$$\begin{aligned}\Delta N_3=(\mathrm{sh}\,\zeta+\zeta\,\mathrm{ch}\,\zeta)[\eta\,\mathrm{sh}\,\beta\,\mathrm{ch}\,(\beta+\eta)-\beta\eta\,\mathrm{ch}\,\eta+\mathrm{sh}\,\eta(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta+\beta^2-\beta)]\\+\zeta\,\mathrm{sh}\,\zeta[\beta(\beta+\eta)\,\mathrm{sh}\,\eta+\eta\,\mathrm{sh}\,\beta\,\mathrm{sh}\,(\beta+\eta)]+\frac{1}{2}\Delta\,\mathrm{e}^{\eta}[(1+\zeta)\,\mathrm{e}^{\zeta}+\eta\,\mathrm{e}^{-\zeta}]\end{aligned} \tag{41c}$$

$$\begin{aligned}\Delta N_4=(\mathrm{sh}\,\zeta+\zeta\,\mathrm{ch}\,\zeta)[\eta\,\mathrm{sh}\,\beta\,\mathrm{sh}\,(\beta+\eta)-\beta(\beta+\eta)\,\mathrm{sh}\,\eta]\\+\zeta\,\mathrm{sh}\,\zeta[\eta\,\mathrm{sh}\,\beta\,\mathrm{ch}\,(\beta+\eta)+\beta\eta\,\mathrm{ch}\,\eta-\mathrm{sh}\,\eta(\mathrm{e}^{-\beta}\,\mathrm{sh}\,\beta+\beta^2+\beta)]\\+\frac{1}{2}\Delta\,\mathrm{e}^{\eta}[(\eta\,\mathrm{e}^{-\zeta}-\zeta\,\mathrm{e}^{\zeta})]\end{aligned} \tag{41d}$$

and

$$\eta=\xi t. \tag{42}$$

$$\frac{\partial u_x}{\partial x}=\frac{1}{2\pi(1+\kappa)}\left\{\int_{-a}^{-b} A(t)t[(\kappa-1)t^2-(3\kappa+1)(x+d)^2]\Delta_3\,dt\right.$$
$$+(x+d)\int_{-a}^{-b} B(t)[(5\kappa-1)t^2+(\kappa-1)(x+d)^2]\Delta_3\,dt$$
$$+\int_{-a}^{-b} C(t)t[(\kappa-1)t^2-(3\kappa+1)(x-d)^2]\Delta_4\,dt$$
$$\left.+(x-d)\int_{-a}^{-b} D(t)[(5\kappa-1)t^2-(\kappa-1)(x-d)^2]\Delta_4\,dt\right\}$$
$$+\frac{1}{\pi}\int_{-a}^{-b}\left[A(t)\int_0^\infty N_5\cos\xi(x+d)\,d\xi+B(t)\int_0^\infty N_6\sin\xi(x+d)\,d\xi\right.$$
$$\left.+C(t)\int_0^\infty N_5\cos\xi(x-d)\,d\xi+D(t)\int_0^\infty N_6\sin\xi(x-d)\,d\xi\right]dt \tag{43a}$$

$$\frac{\partial u_y}{\partial x}=\frac{1}{2\pi(1+\kappa)}\left\{(x+d)\int_{-a}^{-b} A(t)[(3\kappa+1)t^2-(\kappa-1)(x+d)^2]\Delta_3\,dt\right.$$
$$+\int_{-a}^{-b} B(t)t[-(3\kappa+1)t^2+(\kappa-1)(x+d)^2]\Delta_3\,dt$$
$$+(x-d)\int_{-a}^{-b} C(t)[(3\kappa+1)t^2-(\kappa-1)(x-d)^2]\Delta_4\,dt$$
$$\left.+\int_{-a}^{-b} D(t)t[-(3\kappa+1)t^2+(\kappa-1)(x-d)^2]\Delta_4\,dt\right\}$$
$$+\frac{1}{\pi}\int_{-a}^{-b}\left[A(t)\int_0^\infty N_7\sin\xi(x+d)\,d\xi+B(t)\int_0^\infty N_8\cos\xi(x+d)\,d\xi\right.$$
$$\left.+C(t)\int_0^\infty N_7\sin\xi(x-d)\,d\xi+D(t)\int_0^\infty N_8\cos\xi(x-d)\,d\xi\right]dt \tag{43b}$$

where

$$\Delta_4=\Delta_2(t,x,0) \tag{44}$$

$$\left.\begin{aligned}
\Delta N_5&=\eta[e^{-\beta}\,\mathrm{sh}\,\beta-\beta)\,\mathrm{ch}\,\eta+e^{\eta}\beta^2]+\mathrm{sh}\,\eta(e^{-\beta}\,\mathrm{sh}\,\beta+\beta^2-\beta) &(45a)\\
\Delta N_6&=\eta[(\beta-e^{-\beta}\,\mathrm{sh}\,\beta)\,\mathrm{sh}\,\eta-e^{\eta}\beta^2]+\beta^2\,\mathrm{sh}\,\eta &(45b)\\
\Delta N_7&=-\eta[(e^{-\beta}\,\mathrm{sh}\,\beta+\beta)\,\mathrm{sh}\,\eta+e^{\eta}\beta^2]-\beta^2\,\mathrm{sh}\,\eta &(45c)\\
\Delta N_8&=-\eta[(e^{-\beta}\,\mathrm{sh}\,\beta+\beta)\,\mathrm{ch}\,\eta+e^{\eta}\beta^2]+\mathrm{sh}\,\eta(e^{-\beta}\,\mathrm{sh}\,\beta+\beta^2+\beta) &(45d)
\end{aligned}\right\}.$$

The superposition of the two crack solutions and the layer solution leads to the required solution for the cracked layer shown in Fig. 1c.

The boundary conditions, eqns (18) and (19), lead to the following singular integral equations:

$$p_1(y)=\frac{2\mu}{\pi(1+\kappa)}\int_{-a}^{-b}\left\{A(t)\left[-1/(t-y)+2\int_0^\infty N_1\,d\xi\right]\right.$$
$$+C(t)[-\Delta_2(t,-d,y)(t-y)[(t-y)^2+12d^2]$$
$$+2\int_0^\infty N_1\cos(2\xi d)\,d\xi]+D(t)[-2d\Delta_2(t,-d,y)[(t-y)^2-4d^2]$$
$$\left.-2\int_0^\infty N_2\sin(2\xi d)\,d\xi]\right\}dt,\qquad -a<y<-b \tag{46a}$$

$$q_1(y)=\frac{2\mu}{\pi(1+\kappa)}\int_{-a}^{-b}\left\{B(t)\left[-1/(t-y)+2\int_0^\infty N_4\,d\xi\right]\right.$$
$$+C(t)\left[-2d\Delta_2(t,-d,y)[(t-y)^2+4d^2]-2\int_0^\infty N_3\sin(2\xi d)\,d\xi\right]$$
$$+D(t)\left[-\Delta_2(t,-d,y)(t-y)[(t-y)^2-4d^2]\right.$$

$$+2\int_0^\infty N_4\cos(2\xi d)\,d\xi\Big]\Big\}\,dt,\qquad -a<y<-b \tag{46b}$$

$$p_2(y)=\frac{2\mu}{\pi(1+\kappa)}\int_{-a}^{-b}\Big\{C(t)\Big[-1/(t-y)+2\int_0^\infty N_1\,d\xi\Big]$$
$$+A(t)\Big[-\Delta_1(t,d,y)(t-y)[(t-y)^2+12d^2]$$
$$+2\int_0^\infty N_1\cos(2\xi d)\,d\xi\Big]+B(t)\Big[2d\Delta_1(t,d,y)[(t-y)^2-4d^2]$$
$$+2\int_0^\infty N_2\sin(2\xi d)\,d\xi\Big]\Big\}\,dt,\qquad -a<y<-b \tag{46c}$$

$$q_2(y)=\frac{2\mu}{\pi(1+\kappa)}\int_{-a}^{-b}\Big\{D(t)\Big[-1/(t-y)+2\int_0^\infty N_4\,d\xi\Big]$$
$$+A(t)\Big[2d\Delta_1(t,d,y)[(t-y)^2-4d^2]+2\int_0^\infty N_3\sin(2\xi d)\,d\xi\Big]$$
$$+B(t)\Big[-\Delta_1(t,d,y)(t-y)[(t-y)^2-4d^2]$$
$$+2\int_0^\infty N_4\cos(2\xi d)\,d\xi\Big]\Big\}\,dt,\qquad -a<y<-b. \tag{46d}$$

The differentiation of eqns (20) with respect to x leads to

$$\left(\frac{\kappa_1+1}{4\mu_1}\right)\frac{dg_1}{dx}=\frac{1}{\pi}\int_{-a}^{-b}\Big\{A(t)\Big[-2(x+d)^2\Delta_3 t+\int_0^\infty N_5\cos\xi(x+d)\,d\xi\Big]+B(t)\Big[2(x+d)\Delta_3 t^2$$
$$+\int_0^\infty N_6\sin\xi(x+d)\,d\xi\Big]+C(t)\Big[-2(x-d)^2\Delta_4 t+\int_0^\infty N_5\cos\xi(x-d)\,d\xi\Big]$$
$$+D(t)\Big[2(x-d)\Delta_4 t^3+\int_0^\infty N_6\sin\xi(x-d)\,d\xi\Big]\Big\}\,dt \tag{47a}$$

$$\left(\frac{\kappa_1+1}{4\mu_1}\right)\frac{dg_2}{dx}=\frac{1}{\pi}\int_{-a}^{-b}\Big\{A(t)\Big[2(x+d)\Delta_3 t^2+\int_0^\infty N_7\sin\xi(x+d)\,d\xi\Big]$$
$$+B(t)\Big[-2\Delta_3 t^3+\int_0^\infty N_8\cos\xi(x+d)\,d\xi\Big]+C(t)\Big[2(x-d)\Delta_4 t^2$$
$$+\int_0^\infty N_7\sin\xi(x-d)\,d\xi\Big]+D(t)\Big[-2\Delta_4 t^3+\int_0^\infty N_8\cos\xi(x+d)\,d\xi\Big]\Big\}\,dt. \tag{47b}$$

Superposition of Problems I and II

The governing singular integral equations are obtained from eqns (21) to (23) by letting b equal zero and a equal h. They are written in the form

$$\frac{1}{\pi}\int_{-c}^{c}[\tau(s)M_1(s,-d,y)+\sigma(s)M_2(s,-d,y)]\,ds+\lim p_1(y)=-p,-h<y<0 \tag{48a}$$

$$\frac{1}{\pi}\int_{-c}^{c}[\tau(s)M_3(s,-d,y)+\sigma(s)M_4(s,-d,y)]\,ds+\lim q_1(y)=0,-h<y<0 \tag{48b}$$

$$\frac{1}{\pi}\int_{-c}^{c}[\tau(s)M_1(s,d,y)+\sigma(s)M_2(s,d,y)]\,ds+\lim p_2(y)=0,-h<y<0 \tag{48c}$$

$$\frac{1}{\pi}\int_{-c}^{c}[\tau(s)M_3(s,d,y)+\sigma(s)M_4(s,d,y)]\,ds+\lim q_2(y)=0,-h<y<0 \tag{48d}$$

where the notation

$$\lim=\lim_{\substack{a\to h\\ b\to 0}}$$

is used.

In the equations given above, the kernels M_1, M_2, M_3 and M_4 are defined by eqns (32) and the stresses p_1, q_1, p_2 and q_2 by eqns (46). Equations (48) represent the boundary conditions for stresses on the surfaces $x = \pm d$ of the layer. The continuity conditions for displacements can be obtained by differentiating eqns (23) with respect to x and substituting eqns (24) and (47) into the result. This gives two singular integral equations of the form

$$\frac{1}{\pi}\int_{-c}^{c} \tau(s)\left[\frac{1}{s-x} + \pi K_1(s,x)\right] \mathrm{d}s + \alpha_1\sigma(x) + \int_{-c}^{c} \sigma(s)K_2(s,x)\,\mathrm{d}s + \frac{(1+\alpha_2)}{2}\lim\frac{\mathrm{d}g_1}{\mathrm{d}x} = 0, \qquad 0 \le |x| \le c, \tag{49a}$$

$$\frac{1}{\pi}\int_{-c}^{c} \sigma(s)\left[\frac{1}{s-x} + \pi K_3(s,x)\right] \mathrm{d}s - \alpha_1\tau(x) - \int_{-c}^{c} \tau(s)K_2(s,x)\,\mathrm{d}s + \frac{(1+\alpha_2)}{2}\lim\frac{\mathrm{d}g_2}{\mathrm{d}x} = 0, \qquad 0 \le |x| \le c. \tag{49b}$$

Equations (48) provide a set of singular integral equations to be solved for the unknown quantities which, in this case, are the bond stresses and the slopes on the surfaces $|x| = d$. In addition to the singular integral equations, the static equilibrium of the bond stresses, eqns (29) must be satisfied. It should be noted that there are usually consistency equations associated with the cracks located on $|x| = d$. However, the nature of the singularities, which is discussed later, will not require these conditions to be imposed.

Nature of singularities

For the case when $d > c$, the unknown functions have integrable singularities at their end points which are square root and will be written in the following form:

$$\begin{aligned} \tau(s) &= \tau_0(s)(c^2 - s^2)^{-1/2} \\ \sigma(s) &= \sigma_0(s)(c^2 - s^2)^{-1/2} \\ A(t) &= A_0(t)t^{1/2}(t+h)^{1/2} \\ &\cdots\cdots \\ D(t) &= D_0(t)t^{1/2}(t+h)^{1/2}. \end{aligned} \tag{50}$$

In eqns (50) the choice of singularity at $s = \pm c$ for $\tau(s)$, $\sigma(s)$ is obviously square root, since if the layer and half-space are of identical materials, its behavior should be the same as that for an external crack. However, the nature of the singularity for A, B, C, D at $t = 0$, h is not in its correct form as given by eqns (50). But the collocation scheme to be used later is conjectured to give the significant stress intensity factors at $s = \pm c$ with sufficient accuracy as well as other physical quantities of interest. This assertion will be discussed in the next section.

When $d = c$, the kernels in eqns (48) and (49) become unbounded when $s = \pm c$ and $y = 0$ and the integral equations become generalized singular integral equations [11]. By separating out the singular parts from each equation in the system of integral equations conditions appropriate for the corners can be established. By assuming solution forms as follows;

$$\begin{aligned} \tau(s) &= \tau_0(s)(c^2 - s^2)^{\lambda-1} \\ \sigma(s) &= \sigma_0(s)(c^2 - s^2)^{\lambda-1} \\ A(t) &= A_0(t)t^{\lambda-1}(t+h)^{1/2} \\ &\cdots\cdots \\ D(t) &= D_0(t)t^{\lambda-1}(t+h)^{1/2} \end{aligned} \tag{51}$$

where λ is a positive constant, $0 < \lambda < 1$, the following corner conditions result for $s \to -c$, $y \to 0$:

$$\Gamma_1(1+\lambda)\sin\left(\frac{1}{2}\pi\lambda\right) + \Gamma_2\lambda\cos\left(\frac{1}{2}\pi\lambda\right) + A_0(0)\left[\sin^2\left(\frac{1}{2}\pi\lambda\right) - \lambda^2\right] = 0 \tag{52a}$$

$$\Gamma_1\lambda\cos\left(\frac{1}{2}\pi\lambda\right) + \Gamma_2(1-\lambda)\sin\left(\frac{1}{2}\pi\lambda\right) + B_0(0)\left[\sin^2\left(\frac{1}{2}\pi\lambda\right) - \lambda^2\right] = 0 \tag{52b}$$

$$-2\Gamma_1 \cos(\pi\lambda) + A_0(0)(1-\lambda)\sin\left(\frac{1}{2}\pi\lambda\right) + B_0(0)\lambda\cos\left(\frac{1}{2}\pi\lambda\right) = 0 \tag{52c}$$

$$-2\Gamma_2 \cos(\pi\lambda) + A_0(0)\lambda\cos(\pi\lambda) + B_0(0)(1+\lambda)\sin\left(\frac{1}{2}\pi\lambda\right) = 0 \tag{52d}$$

where

$$\Gamma_1 = \tau_0(-c)(2c)^{\lambda-1}/h^{1/2} \tag{53a}$$

$$\Gamma_2 = \sigma_0(-c)(2c)^{\lambda-1}/h^{1/2}. \tag{53b}$$

Similarly, the following conditions result for $s \to c$, $y \to 0$:

$$-\Gamma_3(1+\lambda)\sin\left(\frac{1}{2}\pi\lambda\right) + \Gamma_4\lambda\cos\left(\frac{1}{2}\pi\lambda\right) + C_0(0)\left[\sin^2\left(\frac{1}{2}\pi\lambda\right) - \lambda^2\right] = 0 \tag{54a}$$

$$\Gamma_3\lambda\cos\left(\frac{1}{2}\pi\lambda\right) - \Gamma_4(1-\lambda)\sin\left(\frac{1}{2}\pi\lambda\right) + D_0(0)\left[\sin^2\left(\frac{1}{2}\pi\lambda\right) - \lambda^2\right] = 0 \tag{54b}$$

$$2\Gamma_3\cos(\pi\lambda) + C_0(0)(1-\lambda)\sin\left(\frac{1}{2}\pi\lambda\right) - D_0(0)\lambda\cos\left(\frac{1}{2}\pi\lambda\right) = 0 \tag{54c}$$

$$2\Gamma_4\cos(\pi\lambda) - C_0(0)\lambda\cos\left(\frac{1}{2}\pi\lambda\right) + D_0(0)(1+\lambda)\sin\left(\frac{1}{2}\pi\lambda\right) = 0 \tag{54d}$$

where

$$\Gamma_3 = \tau_0(c)(2c)^{\lambda-1}/h^{1/2} \tag{55a}$$

$$\Gamma_4 = \sigma_0(c)(2c)^{\lambda-1}/h^{1/2}. \tag{55b}$$

A nontrivial solution to eqns (52) and (54) is obtained only when the determinant of the coefficients is equal to zero. Both sets of equations give

$$[\sin^2(3\pi\lambda/2) - \lambda^2][\sin^2(\pi\lambda/2) - \lambda^2] = 0. \tag{56}$$

It is seen from the above equation that the value of λ obtained from the equation

$$\sin(3\pi\lambda/2) - \lambda = 0 \qquad (\lambda = 0{\cdot}54448) \tag{57}$$

will provide the required singularity in agreement with Williams [12] for a right angle corner. It is noted that the equation

$$\sin(3\pi\lambda_1/2) + \lambda_1 = 0$$

also gives a singularity for $0 < \lambda_1 < 1$ but in this case $\lambda_1 > \lambda$ and is not the dominant one.

Numerical procedure and results

The numerical solution is divided into two parts according to the nature of the singularities of the unknown functions. When $d > c$, the unknown functions take the form given by eqns (50) and can be solved by using the singular integral equations, eqns (48) and (49), together with the static equilibrium conditions, eqns (29). Numerical results will be given only for the case of identical materials for the layer and half-space. The parameters used in the calculation are d/c and c/h, which are geometrical. When $d = c$ and the materials are identical and the only parameter is c/h, the unknown functions have the form given by eqns (51) with the positive constant λ being determined from eqn (57). In this case eqns (48) and (49) and eqns (29) are used along with the corner conditions, eqns (52) and (54). Since eqns (52) and (54) are eigenvalue problems, only three of the four equations are necessary from each group.

In eqns (50) and (51) singularities at $y = -h$ are assumed to be of $0(t^{1/2})$ as $t \to -h$, which is not the correct one. These singularities are introduced as a convenience to be used with the numerical scheme that follows. Since they are not of the correct form, one questions whether significant accuracy is lost by using such singularities for the calculation of physical quantities. To test the accuracy of the method the edge crack problem of Sneddon and Das[12] was formulated as a singular integral equation. At the tip of the crack the singularity was $0[(c-t)^{-1/2}]$ as $t \to c$ while at the free surface the crack was given a singularity of $0(t^{1/2})$ as $t \to 0$. The method of Erdogan, Gupta and Cook[11] was used to numerically solve the integral equation. The comparison of physical quantities (stress intensity factor and strain energy) with their results was sufficiently close as to be considered exact. Below are shown results from the present analysis compared with the most accurate results given by Sneddon and Das:

	Present Analysis	Sneddon and Das
Stress intensity factor	0·7929	0·7930
Strain energy	0·9877	0·9877

It is convenient for the numerical analysis to introduce the following changes of variables in eqns (48) and (49):

$$s = cs_0, \qquad x = cx_0$$
$$t = ht_0 = \frac{1}{2}h(t_0' - 1), \qquad y = hy_0 = \frac{1}{2}h(y_0' - 1) \tag{58}$$
$$c = \alpha h, \qquad d = \delta h$$

and

$$\tau(s) = \tau(cs_0) = p\phi_1(s_0)(1 - s_0^2)^{\lambda_1 - 1}$$
$$\sigma(s) = \sigma(cs_0) = p\phi_2(s_0)(1 - s_0^2)^{\lambda_1 - 1} \tag{59}$$
$$A(t) = A(ht_0) = \frac{p(1+\kappa)}{4\mu} A_0(t_0')(1 - t_0')^{\lambda_2 - 1}(1 + t_0')^{1/2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$D(t) = D(ht_0) = \frac{p(1+\kappa)}{4\mu} D_0(t_0')(1 - t_0')^{\lambda_2 - 1}(1 + t_0')^{1/2}$$

where $\lambda_1 = 1/2$, $\lambda_2 = 3/2$ for the case $d > c$ and $\lambda_1 = \lambda_2 = \lambda$ for the case $d = c$.

Substituting eqns (58) and (59) into eqns (48) and (49) and using a Gaussian quadrature formula to evaluate the integrals for appropriately selected values of the variables lead to the following system of simultaneous algebraic equations:

$$\frac{\alpha}{\pi}\sum_{k=1}^{n} C_k[\phi_1(s_k)hM_1(cs_k, -\delta h, hy_r) + \phi_2(s_k)hM_2(cs_k, -\delta h, hy_r)]$$
$$+ \frac{1}{2\pi}\sum_{k=1}^{n} D_k\Big\{A_0(t_k')\Big[-1/(t_k - y_r) + 2\int_0^{\infty} N_1\,\mathrm{d}\beta\Big]$$
$$+ C_0(t_k')[-\Delta_2(ht_k, -\delta h, hy_r)^3(t_k - y_r)[(t_k - y_r)^2 + 12\delta^2]$$
$$+ 2\int_0^{\infty} N_1\cos(2\beta\delta)\,\mathrm{d}\beta\Big] + D_0(t_k')\Big[-2\delta\Delta_2(ht_k, -\delta h, hy_r)h^3[(t_k - y_r)^2 + 12\delta^2]$$
$$- 2\int_0^{\infty} N_2\sin(2\beta\delta)\,\mathrm{d}\beta\Big]\Big\} = -1 \tag{60a}$$

$$-\sum_{k=1}^{n} C_k[\phi_1(s_k)hM_3(cs_k, -\delta h, hy_r) + \phi_2(s_k)hM_4(cs_k, -\delta h, hy_r)]$$
$$+ \frac{1}{2\pi}\sum_{k=1}^{n} D_k\Big\{B_0(t_k')\Big[-1/(t_k - y_r) + 2\int_0^{\infty} N_4\,\mathrm{d}\beta\Big]$$

$$+C_0(t_k')\left[-2\delta\Delta_2(ht_k,-\delta h,hy_r)h^3[(t_k-y_r)^2-4\delta^2]+2\int_0^\infty N_3\sin(2\beta\delta)\,\mathrm{d}\beta\right]$$
$$+D_0(t_k')[-\Delta_2(ht_k,-\delta h,hy_r)h^3(t_k-y_r)[(t_k-y_r)^2-4\delta^2]$$
$$+2\int_0^\infty N_4\cos(2\beta\delta)\,d\beta\Big]\Big\}=0 \qquad (60b)$$

$$-\sum_{k=1}^{n}C_k[\phi_1(s_k)hM_1(cs_k,\delta h,hy_r)+\phi_2(s_k)hM_2(cs_k,\delta h,hy_r)]$$
$$+\frac{1}{2\pi}\sum_{k=1}^{n}D_k\Big\{C_0(t_k')\left[-1/(t_k-y_r)+2\int_0^\infty N_1\,\mathrm{d}\beta\right]$$
$$+A_0(t_k')\Big[-\Delta_1(ht_k,\delta h,hy_r)h^3(t_k-y_r)[(t_k-y_r)^2+12\delta^2]$$
$$+2\int_0^\infty N_1\cos(2\beta\delta)\,\mathrm{d}\beta\Big]+B_0(t_k')\Big[2\delta\Delta_1(ht_k,\delta h,hy_r)h^3[(t_k-y_r)^2-4\delta^2]$$
$$+2\int_0^\infty N_2\sin(2\beta\delta)\,\mathrm{d}\beta\Big]\Big\}=0 \qquad (60c)$$

$$\sum_{k=1}^{n}C_k[\phi_1(s_k)hM_3(cs_k,\delta h,hy_r)+\phi_2(s_k)hM_4(cs_k,\delta h,hy_r)]$$
$$+\frac{1}{2\pi}\sum_{k=1}^{n}D_k\Big\{D_0(t_k')\left[-1/(t_k-y_r)+2\int_0^\infty N_4\,\mathrm{d}\beta\right]$$
$$+A_0(t_k')\Big[2\delta\Delta_1(ht_k,\delta h,hy_r)h^3[(t_k-y_r)^2-4\delta^2]$$
$$+2\int_0^\infty N_3\sin(2\beta\delta)\,\mathrm{d}\beta\Big]+B_0(t_k')\Big[-\Delta_1(ht_k,\delta h,hy_r)h^3(t_k-y_r)[(t_k-y_r)^2-4\delta^2]$$
$$+2\int_0^\infty N_4\cos(2\beta\delta)\,\mathrm{d}\beta\Big]\Big\}=0 \qquad (60d)$$

$$\sum_{k=1}^{n}C_k[\phi_1(s_k)[2/(s_k-x_r)+\alpha hK_1(cs_k,cx_r)]+\alpha hK_2(cs_k,cx_r)\phi_2(s_k)]$$
$$+\sum_{k=1}^{n}D_k\Big\{A_0(t_k')\left[-2(\alpha x_r+\delta)^2h^3\Delta_3t_k+\int_0^\infty N_5\cos\beta(\alpha x_r+\delta)\,\mathrm{d}\beta\right]$$
$$+B_0(t_k')\left[2(\alpha x_r+\delta)h^3\Delta_3t_k^2+\int_0^\infty N_6\sin\beta(\alpha x_r+\delta)\,\mathrm{d}\beta\right]$$
$$+C_0(t_k')\left[-2(\alpha x_r-\delta)^2h^3\Delta_6t_k+\int_0^\infty N_5\cos\beta(\alpha x_r-\delta)\,\mathrm{d}\beta\right]$$
$$+D_0(t_k')\left[2(\alpha x_r-\delta)h^3\Delta_6t_k^2+\int_0^\infty N_6\sin\beta(\alpha x_r-\delta)\,\mathrm{d}\beta\right]\Big\}=0 \qquad (60e)$$

$$\sum_{k=1}^{n}C_k[\phi_2(s_k)[2/(s_k-x_r)+\alpha hK_3(cs_k,cx_r)]-\alpha hK_3(cs_k,cx_r)\phi_1(s_k)]$$
$$+\sum_{k=1}^{n}D_k\Big\{A_0(t_k')\left[2(\alpha x_r+\delta)h^3\Delta_3t_k^2+\int_0^\infty N_7\sin\beta(\alpha x_r+\delta)\,\mathrm{d}\beta\right]$$
$$+B_0(t_k')\left[-2h^3\Delta_3t_k^3+\int_0^\infty N_8\cos\beta(\alpha x_r+\delta)\,\mathrm{d}\beta\right]$$
$$+C_0(t_k')\left[2(\alpha x_r-\delta)h^3\Delta_6t_k^2+\int_0^\infty N_7\sin\beta(\alpha x_r-\delta)\,\mathrm{d}\beta\right]$$
$$+D_0(t_k')\left[-2h^3\Delta_6t_k^3+\int_0^\infty N_8\cos\beta(\alpha x_r-\delta)\,\mathrm{d}\beta\right]\Big\}=0 \qquad (60f)$$

where the collocation points, s_k, x_r, t_k' and y_r', are the roots of the following Jacobi polynomial equations:

$$\left.\begin{array}{r} P_n^{(\lambda_1-1,\lambda_1-1)}(s_k)=0 \\ P_{n-1}^{(\lambda_1,\lambda_1)}(x_r)=0 \end{array}\right\} \tag{61}$$

$$\left.\begin{array}{r} P_n^{(1/2,1/2)}(t'_k)=0 \\ P_{n+1}^{(-1/2,-1/2)}(y'_r)=0 \end{array}\right\}, \quad d>c \tag{62a}$$

$$\left.\begin{array}{r} P_n^{(\lambda_2-1,1/2)}(t'_k)=0 \\ P_n^{(\lambda_2,-1/2)}(y'_r)=0 \end{array}\right\}, \quad d=c \tag{62b}$$

and

$$t_k=(t'_k-1)/2 \qquad y_r=(y'_r-1)/2. \tag{63}$$

The coefficients C_k and D_k are the corresponding weights of eqns (62), (62a) or (62b).

The application of eqns (58) and (59) into the subsidiary conditions, eqns (29), (52) and (54), leads to the following equations:

$$\sum_{k=1}^{n} C_k\phi_1(s_k)=-1/\alpha \tag{64a}$$

$$\sum_{k=1}^{n} C_k\phi_2(s_k)=0 \tag{64b}$$

$$\Gamma'_1(1+\lambda)\sin\left(\frac{1}{2}\pi\lambda\right)+\Gamma'_2\lambda\cos\left(\frac{1}{2}\pi\lambda\right)+A_0(1)\left[\sin^2\left(\frac{1}{2}\pi\lambda\right)-\lambda^2\right]=0 \tag{65a}$$

$$\Gamma'_1\lambda\cos\left(\frac{1}{2}\pi\lambda\right)+\Gamma'_2(1-\lambda)\sin\left(\frac{1}{2}\pi\lambda\right)+B_0(1)\left[\sin^2\left(\frac{1}{2}\pi\lambda\right)-\lambda^2\right]=0, \tag{65b}$$

$$-2\Gamma'_1\cos(\pi\lambda)+A_0(1)(1-\lambda)\sin\left(\frac{1}{2}\pi\lambda\right)+B_0(1)\cos\left(\frac{1}{2}\pi\lambda\right)=0 \tag{65c}$$

$$-\Gamma'_3(1+\lambda)\sin\left(\frac{1}{2}\pi\lambda\right)+\Gamma'_4\lambda\cos\left(\frac{1}{2}\pi\lambda\right)+C_0(1)\left[\sin^2\left(\frac{1}{2}\pi\lambda\right)-\lambda^2\right]=0 \tag{66a}$$

$$\Gamma'_3\lambda\cos\left(\frac{1}{2}\pi\lambda\right)-\Gamma'_4(1-\lambda)\sin\left(\frac{1}{2}\pi\lambda\right)+D_0(1)\left[\sin^2\left(\frac{1}{2}\pi\lambda\right)-\lambda^2\right]=0, \tag{66b}$$

$$2\Gamma'_3\cos(\pi\lambda)+C_0(1)(1-\lambda)\sin\left(\frac{1}{2}\pi\lambda\right)-D_0(1)\lambda\cos\left(\frac{1}{2}\pi\lambda\right)=0 \tag{66c}$$

where

$$\Gamma'_i=\phi_i(-1)(2\alpha)^{\lambda-1}, \qquad i=1,2 \tag{67a}$$

$$\Gamma'_i=\phi_i(1)(2\alpha)^{\lambda-1}, \qquad i=3,4. \tag{67b}$$

For the case where $d>c$, a set of $6n\times 6n$ simultaneous algebraic equations is obtained from eqns (60) and eqns (64). It is noted that eqns (62) give $n+1$ values of y'_r that satisfy eqns (60a)–(60d) and therefore four equations corresponding to one of y'_r's must be dropped. In general the equations corresponding to $y'_{n/2+1}$ are ignored. The case where $d=c$ presents some problems in the determination of the unknowns due to the fact that the corner conditions, eqns (65) and (66), must also be satisfied. The simultaneous algebraic equations are solved by first ignoring the six corner equations, eqns (65) and (66), but using appropriate collocation points. Using solutions of the case where $d>c$ as a check, it is observed that the accuracy of the bond stresses obtained in this case are in a reasonable range but that the slopes are not. Next, six equations are dropped from the simultaneous algebraic equations in favor of the six corner conditions. These equations correspond to y'_n and y'_{n-1} (two equations from eqns (60a) and (60b), four equations from eqns (60c) and (60d). The results obtained in this case are in a reasonable range with those previously obtained for the case where $d>c$.

Numerical solutions were obtained for the cases where the ratio of layer length to bond length, d/c, are 1·25 and 1. In each case the geometrical parameter c/h had the values of 0·5, 0·75, 1, 2 and 3. The number of points, n, was taken as 16 in all cases considered. The range of parameters given by the above method appears to be limited. The nature of the bond stresses for large values of c/h seems to require n to be greater than 16 in order to get a good approximation outside of this range. Due to the choice of unknowns in this problem, which are the bond stresses in the bond region, $-c \leq x \leq c$, and the slopes on the surfaces of the layer, $-h \leq y \leq 0$, the number of points used to approximate these unknowns, for a particular value of c/h, should be different for the two regions in order to obtain a good approximation. Otherwise there is difficulty when c/h is small (corresponding to a beam problem). For this case the number of points, n, has to be greater than 16 to give accurate results. Such a large number is not practical in this numerical evaluation due to the amount of unknowns for the problem.

By separating the applied uniform load into symmetric and antisymmetric components, the unknowns for both cases can be obtained. The required solution is the sum of solutions of these two cases. For the symmetric case the problem is solved by employing the method described in [6], where an elementary extension solution is added to the layer to provide an appropriate displacement mismatch in the bond region. (Eqns 11).

The physical quantities of interest are the stress intensity factors, k_1 and k_2, of the shear bond stress and k_3 and k_4 of the normal bond stress. These quantities are defined as follows:

$$k_1 = -\lim_{s_0 \to -1} [2(1+s_0)]^{1-\lambda_1} \tau(cs_0)/p = -\phi_1(-1) \tag{68a}$$

$$k_2 = -\lim_{s_0 \to 1} [2(1-s_0)]^{1-\lambda_1} \tau(cs_0)/p = -\phi_1(1) \tag{68b}$$

$$k_3 = \lim_{s_0 \to -1} [2(1+s_0)]^{1-\lambda_1} \sigma(cs_0)/p = \phi_2(-1) \tag{68c}$$

$$k_4 = \lim_{s_0 \to 1} [2(1-s_0)]^{1-\lambda_1} \sigma(cs_0)/p = \phi_2(1). \tag{68d}$$

In eqns (68) the stresses are given by the definitions, eqns (59) and $\lambda_1 = 1/2$ for the case $d > c$ and $\lambda_1 = \lambda$ for the case $d = c$.

Tables 1 and 2 show the stress intensity factors for the cases where d/c are equal to 1·25 and 1 for various values of c/h. It is observed that the stress intensity factors decrease as c/h increases, as should be the case. Also the range of parameters here is not sufficient for the stress intensity factors to approach a constant value. Table 3 gives the stress intensity factors for the case where $c/h = 1$ for various values of d/c. It is seen that the stress intensity factors approach the values for an infinite strip very quickly as the layer length increases relative to the bond. In

Table 1. Stress intensity factors vs the geometrical parameter α with $d/c = 1{\cdot}25$

c/h	k_1	k_2	k_3	k_4
.5	.648	.458	1.349	-1.309
1	.367	.188	.398	-.325
2	.232	.066	.183	-.076
3	.185	.033	.145	-.033

Table 2. Stress intensity factors vs the geometrical parameter α with $d/c = 1$

c/h	k_1	k_2	k_3	k_4
.5	.736	.623	1.354	-1.147
.75	.458	.342	.842	-.629
1	.326	.220	.601	-.390
2	.164	.058	.302	-.106
3	.116	.025	.214	-.046

Table 3. Stress intensity factors vs the geometrical parameter d/c with $\alpha = 1$

d/c	k_1	k_2	k_3	k_4
1.25	.3671	.1881	.3979	−.3253
1.50	.3845	.1979	.3896	−.3205
2	.3872	.1997	.3895	−.3204

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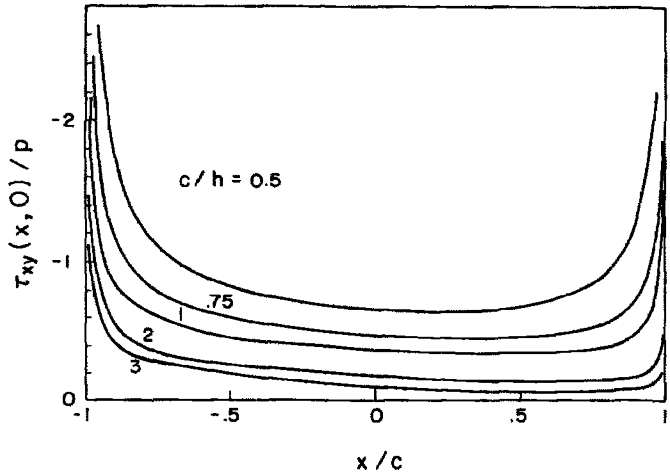

Fig. 5. Shear bond stress vs x/c for $d/c = 1$.

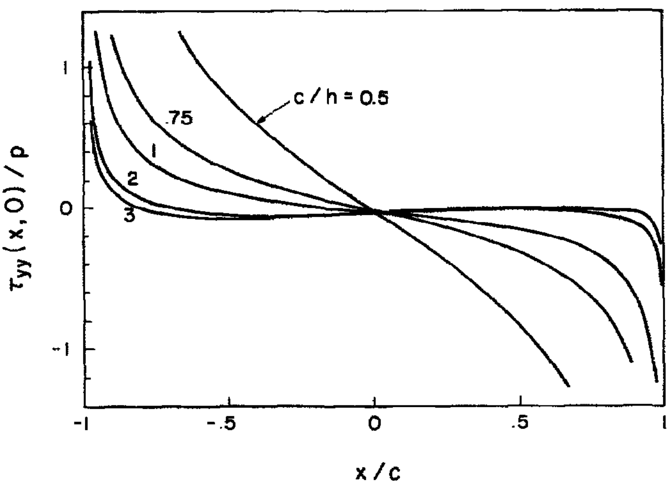

Fig. 6. Normal bond stress vs x/c for $d/c = 1$.

stress intensity factors is such that $k_3 > k_1$ for small values of α and this inequality is reversed for larger values of α. On the other hand from Table 2 it can be seen that in all cases the value of k_3 is greater than that of k_1 by about a factor of two. Thus, there is a qualitative difference in the ratios of shear and normal stress intensity factors.

Acknowledgement—The authors are grateful for partial support of this research from the National Science Foundation under Grant ENG72-04195 A01. A portion of the work was completed by one of the authors (L.M.K.) while a Guggenheim Fellow at the Department of Mathematics, University of Glasgow.

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